

DISORDERED FERMIONS ON LATTICES AND THEIR SPECTRAL PROPERTIES

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ABSTRACT. We study Fermionic systems on a lattice with random interactions through their dynamics and the associated KMS states. These systems require a more complex approach compared with the standard spin systems on a lattice, on account of the difference in commutation rules for the local algebras for disjoint regions, between these two systems. It is for this reason that some of the known formulations and proofs in the case of the spin lattice systems with random interactions do not automatically go over to the case of disordered Fermion lattice systems. We extend to the disordered CAR algebra, some standard results concerning the spectral properties exhibited by temperature states for disordered quantum spin systems. We discuss the Arveson spectrum and its connection with the Connes and Borchers Γ -invariants for such W^* -dynamical systems. In the case of KMS states exhibiting a natural property of invariance with respect to the spatial translations, some interesting properties, associated with standard spin-glass-like behaviour, emerge naturally. It covers infinite-volume limits of finite-volume Gibbs states, that is the quenched disorder for Fermions living on a standard lattice \mathbb{Z}^d . In particular, we show that a temperature state of the systems under consideration can generate only a type III von Neumann algebra (with the type III_0 component excluded). Moreover, in the case of the pure thermodynamic phase, the associated von Neumann is of type III_λ for some $\lambda \in (0, 1]$, independent of the disorder. Such a result is in accordance with the principle of self-averaging which affirms that the physically relevant quantities do not depend on the disorder. The present approach can be viewed as a further step towards fully understanding the very complicated structure of the set of temperature states of quantum spin glasses, and its connection with the breakdown of the symmetry for the replicas.

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1. INTRODUCTION

Interacting Fermion systems on a lattice have usually been studied by considering spinless Fermions at each lattice site which interact with each other. The restriction to spinless particles is just a matter of simplification of notation and more general situations can be treated as well. Investigations concerning the existence of dynamics have been made in the past and, more recently, the equilibrium statistical mechanics of such systems including the thermodynamic limits have been studied. We refer the reader to [6, 7, 37, 38] and the literature cited therein, for a systematic treatment of the topic.

An example of a Fermion lattice system is the Hubbard model (see e.g. [56, 57]) which describes electrons in a solid, interacting with each other through a repulsive Coulomb force. The spinless counterpart was considered in [39], and some particular case of its disordered version is analyzed in some detail in Section 7 of the present paper. Other (non disordered) models based on Fermions, and connected with the Quantum Markov Property and the Quantum Information Theory, are considered in [1, 24]. Importantly, in the majority of the cases the interaction potential is assumed to be even. Of course, there are situations wherein the potential considered is non even (cf. [5]), but it is yet unclear if non even interactions have relevant physical applications.

Another very important line of research in Statistical Mechanics is that involving the so called *spin glasses*, falling into the more general category of disordered systems. The first model, constructed on tensor product of copies of a single algebra of observables localized on the sites in a lattice, is the Sherrington–Kirkpatrick model (cf. [45]). It is a disordered mean field model for which it is meaningless to define the dynamics in the thermodynamic limit. A more realistic model is the so called Edwards–Anderson model (cf. [19]), which can be considered as a disordered generalization of the Ising or ferrimagnetic model, provided the distribution of the coupling constants is one–sides.¹ The investigation of the *quenched disorder* for spin glasses is a fairly formidable task. Among the problems which are still open, we mention the *breakdown of the symmetry for replicas*. For the convenience of the reader, we report the references [12, 21, 22, 25, 26, 27, 40, 41, 42, 54, 58] which are just a sample (far from being complete), of some of the work done on the theory of the spin glasses.²

¹The most interesting situation, corresponding to a spin glass, is when the distribution of the coupling constants is one–sides.

²A very nice explanation of the failure of the Replica Method Solution, in terms of the Moment Problem is given in [55].

It is then natural to undertake the study of the disordered systems for models which include Fermions. For models without Fermions, the study of such disordered systems was firstly carried out in [32], using the standard techniques of Operator Algebras. Apart from the general properties of such disordered systems established in that paper, the problem of the so called *weak Gibbsianess*, that is the appearance of weakly Gibbsian states which are not jointly Gibbsian with respect to the observable variables and the coupling constants taken together, is well explained. The reader is referred to [20, 33] for some concrete example on weak Gibbsianess relative to the classical case. In [8, 9], general properties of temperature (i.e. KMS) states, and their spectral properties were studied in detail. Finally, in [23], the theory of chemical potential is extended to such disordered systems. The reader is referred also to [4, 10] for a good review on the topic.

In our model we consider Fermions on a lattice with random even interactions between the spinless particles located at the lattice sites. It is expected that the spectra of the random evolution group of this infinite Fermion system will exhibit some invariance properties. Besides, the invariant KMS states are also expected to enjoy some nice structural properties. Because of the complex random structure and the (anti)commutation properties of local algebras, the analysis of such a system is a fairly daunting task. Some of the known formulations and proofs for disorderd spin lattice systems do not automatically go over to the disorderd Fermion lattice systems.

By using a standard procedure (cf. e.g. [11]), we start with an appropriate C^* -algebra of observables, that is $\mathfrak{A} := \mathcal{A} \otimes L^\infty(\Omega, \mu)$. In order to encode the Fermions, we consider a separable unital C^* -algebra \mathcal{A} , equipped with a \mathbb{Z}_2 -grading. In particular, in some concrete examples \mathcal{A} will be the CAR algebra $\text{CAR}(\mathbb{Z}^d)$ on the lattice \mathbb{Z}^d . In order to take into account the disorder, the probability space (Ω, μ) is the sample space for the coupling constants, the latter being random variables on it. For such a disordered system, the lattice translations and the time evolution act in a natural way as mutually commuting group actions. The resulting systems fall into the category of so called graded asymptotically Abelian systems. Due to the grading, the study of the spectral properties of such systems is more involved than that of the asymptotically Abelian ones.

After investigating the general properties of the disordered systems (sections 2, 3, 4), in Section 5 we generalize some spectral properties, known for asymptotically Abelian systems, to the \mathbb{Z}_2 -graded models under considerations. Section 6 is devoted to apply such spectral

results obtained in the graded situation, to the study of the structure of the von Neumann algebras generated by \mathbb{Z}_2 -graded asymptotically Abelian dynamical systems. Then we are able to investigate the type of the von Neumann algebras arising from temperature states of the Fermionic systems under consideration. The reader is referred to [3, 9, 18, 28, 35, 46] for the analogous results known for asymptotically Abelian dynamical systems.

The first general result we are able to prove is that a von Neumann algebra with a nontrivial infinite semifinite summand cannot carry an action which is graded asymptotically Abelian w.r.t. the strong topology. Namely, we generalize the corresponding result known for asymptotically Abelian systems. We then pass on to the investigation of the Arveson spectrum and its connection with the Connes and Borchers Γ -invariants for the (non factor) W^* -dynamical systems equipped with a \mathbb{Z}_2 -grading. We apply such results to W^* -dynamical systems generated by the GNS representation of temperature states exhibiting natural equivariance properties with respect to the spatial translations and the time evolution. Then some interesting properties, associated with the standard spin-glass-like behaviour, emerge naturally. The analysis covers the case of KMS states obtained by infinite volume limits of finite-volume Gibbs states, that is the quenched disorder for Fermions living on a standard lattice \mathbb{Z}^d . We mention the fact that a temperature state of such disordered Fermions can generate only a type III von Neumann algebra, with the type III₀ component excluded.

As explained in [9], the natural candidate for the pure thermodynamic phase is when the center $\pi_\varphi(\mathfrak{A})'' \wedge \pi_\varphi(\mathfrak{A})'$ of the GNS representation of a KMS state φ , is "as trivial as possible", that is

$$\pi_\varphi(\mathfrak{A})'' \wedge \pi_\varphi(\mathfrak{A})' \sim L^\infty(\Omega, \mu).$$

Even for disordered systems including Fermions, a consequence of the previously mentioned result is that for a pure thermodynamic phase φ , π_φ generates a III _{λ} von Neumann algebra, for some $\lambda \in (0, 1]$, independent of the disorder. Namely, for the pure thermodynamic phase of the disordered models under consideration, $\pi_\varphi(\mathfrak{A}) \sim M \overline{\otimes} L^\infty(\Omega, \mu)$, where M is the unique type III _{λ} hyperfinite von Neumann factor. Such a result is in accordance with the principle of self-averaging which affirms that the physically relevant quantities do not depend on the disorder. For a nice explanation on the study of the spectral properties and the connected investigation of the type of the factors appearing in Quantum Statistical Mechanics, the reader is referred to the review paper [30] and the literature cited therein. We cite also the paper [36] where

an interesting connection between the modular structure of the algebras of the observables and the statistics of the black holes is established.

The paper ends with a section devoted to the detailed analysis of a concrete model based on a kind of disordered spinless Fermions, which reduces itself to the disordered Hubbard Hamiltonian, provided the distribution of the coupling constants is one-sides.

The symmetry replica breaking is one of the most important open problems in the theory of the spin glasses. As our approach is naturally based on the replicas, one for each value of the coupling constants, we hope that the approach followed in the present paper, as well as that in the previous connected works [8, 9, 23, 32], can be viewed as a significant step towards fully understanding the very complicated structure of the set of temperature states of quantum spin glasses, and its connection with the breakdown of the symmetry for replicas.

2. THE DESCRIPTION OF THE MODEL

In the present paper we deal only with von Neumann algebras with separable preduals unless specified otherwise. Besides, all representations of the involved C^* -algebras are understood to act on separable Hilbert spaces. Finally, all our C^* -algebras have the identity $\mathbf{1}$. Denote by $[a, b] := ab - ba$, $\{a, b\} := ab + ba$, the commutator and anticommutator between elements a, b , respectively.

We start by quickly reviewing the basic properties of the Fermion C^* -algebra $\text{CAR}(\mathbb{Z}^d)$ on a lattice \mathbb{Z}^d . Indeed, let J be any set. The *Canonical Anticommutation Relations* (CAR for short) algebra over J is the C^* -algebra $\text{CAR}(J)$ with the identity $\mathbf{1}$ generated by the set $\{a_j, a_j^\dagger\}_{j \in J}$ (i.e. the Fermi annihilators and creators respectively), and the relations

$$(a_j)^* = a_j^\dagger, \quad \{a_j^\dagger, a_k\} = \delta_{jk} \mathbf{1}, \quad \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0, \quad j, k \in J.$$

On the CAR algebra the parity automorphism Θ acts on the generators as

$$\Theta(a_j) = -a_j, \quad \Theta(a_j^\dagger) = -a_j^\dagger, \quad j \in J,$$

and induces on $\text{CAR}(J)$ a \mathbb{Z}_2 -grading. This grading yields, $\text{CAR}(J) = \text{CAR}(J)_+ \oplus \text{CAR}(J)_-$ where

$$\begin{aligned} \text{CAR}(J)_+ &:= \{a \in \text{CAR}(J) \mid \Theta(a) = a\}, \\ \text{CAR}(J)_- &:= \{a \in \text{CAR}(J) \mid \Theta(a) = -a\}. \end{aligned}$$

Elements in $\text{CAR}(J)_+$ and in $\text{CAR}(J)_-$ are called *even* and *odd*, respectively.

A map $T : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ between C^* -algebras with \mathbb{Z}_2 -gradings Θ_1, Θ_2 is said to be *even* if it is grading-equivariant:

$$T \circ \Theta_1 = \Theta_2 \circ T.$$

The previous definition applied to states $\varphi \in \mathcal{S}(\text{CAR}(J))$ leads to $\varphi \circ \Theta = \varphi$, that is φ is even if it is Θ -invariant.

Let the index set J be countable, then the CAR algebra is isomorphic to the C^* -infinite tensor product of J -copies of $\mathbb{M}_2(\mathbb{C})$:

$$(2.1) \quad \text{CAR}(J) \sim \overline{\bigotimes_J \mathbb{M}_2(\mathbb{C})}^{C^*}.$$

Such an isomorphism is established by the Jordan–Klein–Wigner transformation, see e.g. [53], Exercise XIV. When $J = \mathbb{Z}^d$, the above mentioned isomorphism does not preserve the canonical local properties of the CAR algebra, thus it cannot be used to investigate the local properties of the model.

Thanks to (2.1), $\text{CAR}(J)$ has a unique tracial state τ as the extension of the unique tracial state on $\text{CAR}(I)$, $|I| < +\infty$. Let $J_1 \subset J$ be a finite set and $\varphi \in \mathcal{S}(\text{CAR}(J))$. Then there exists a unique positive element $T \in \text{CAR}(J_1)$ such that $\varphi|_{\text{CAR}(J_1)} = \tau|_{\text{CAR}(J_1)}(T \cdot)$. The element T is called the *adjusted matrix* of $\varphi|_{\text{CAR}(J_1)}$. For the standard applications to quantum statistical mechanics, one also uses the density matrix w.r.t. the unnormalized trace.

Our aim is to investigate disordered models of Fermions on lattices. Our starting point will be the Fermion algebra $\text{CAR}(\mathbb{Z}^d)$ together with a (formal) random Hamiltonian. We denote $\text{CAR}(\Lambda) \subset \text{CAR}(\mathbb{Z}^d)$ the local CAR subalgebra generated by $\{a_j, a_j^\dagger \mid j \in \Lambda\}$. We can then consider a net $\{H_\Lambda(\omega)\}_{\Lambda \subset \mathbb{Z}^d}$, Λ being any finite subsets of \mathbb{Z}^d , of local random Hamiltonians which are even with respect to the parity automorphism Θ , which is made up of $\text{CAR}(\mathbb{Z}^d)_{s.a.}$ -valued measurable functions arising from finite-range even interactions. The net under consideration satisfies the equivariance condition

$$(2.2) \quad H_{\Lambda+x}(\omega) = \alpha_x(H_\Lambda(T_{-x}\omega)).$$

Such a picture arises naturally in the study of disordered systems (see e.g. [11, 32]), and more precisely when one considers Fermion systems with random, even Hamiltonians. A concrete model arising from a random Hamiltonian as in (2.2) is described in some detail in Section 7.

Associated with such a random Hamiltonian, there is a one parameter group of random automorphisms τ_t^ω of $\text{CAR}(\mathbb{Z}^d)$, one for each choice

of the coupling constants in the sample space Ω .³ As described below, we assume that τ_t^ω enjoys good joint measurability conditions, and local properties, see Section 3. In view of the possible applications to more general situations including disordered gauge theories on the lattices and/or disordered theories arising from quantum field theory, our main object will be merely a unital \mathbb{Z}_2 -graded separable C^* -algebra \mathcal{A} . In Section 7 we specialize to the case of a concrete model for which $\mathcal{A} = \text{CAR}(\mathbb{Z}^d)$.

3. THE DISORDERED ALGEBRA OF THE OBSERVABLES

Taking the cue from the concrete model described in the Section 7, in order to describe disordered models including Fermions on lattices we list all our assumptions.

We start with a separable C^* -algebra \mathcal{A} with an identity $\mathbf{1}$, describing the physical observables/fields.⁴

We suppose that the spatial translations \mathbb{Z}^d acts in a natural way on \mathcal{A} as a group of automorphisms $\{\alpha_x\}_{x \in \mathbb{Z}^d} \subset \text{Aut}(\mathcal{A})$. In addition, we suppose that there exists an automorphism $\sigma \in \text{Aut}(\mathcal{A})$ whose square is the identity, commuting with the spatial translations,

$$(3.1) \quad \sigma^2 = \text{id}; \quad \sigma \alpha_x = \alpha_x \sigma, \quad x \in \mathbb{Z}^d.$$

In the concrete situation when $\mathcal{A} = \text{CAR}(\mathbb{Z}^d)$, α_x is the shift on the lattice of the creators and annihilators by an amount $x \in \mathbb{Z}^d$, whereas σ is nothing but the parity automorphism Θ . We put

$$(3.2) \quad \mathcal{A}_+ := \frac{1}{2}(e + \sigma)(\mathcal{A}), \quad \mathcal{A}_- := \frac{1}{2}(e - \sigma)(\mathcal{A}).$$

Denote by

$$(3.3) \quad \{A, B\}_\epsilon := AB - \epsilon_{A,B}BA$$

the *graded commutator*, where $\epsilon_{A,B} = -1$ if $A, B \in \mathcal{A}_-$ and $\epsilon_{A,B} = 1$ in the case of the three remaining possibilities.

³The reader is referred to the seminal paper [6] concerning the statistical mechanics associated with (non disordered) Fermions.

⁴In the case of gauge theories, \mathcal{A} is obtained as the fixed-point algebra $\mathcal{A} = \mathcal{F}^G$ under a pointwise action $\gamma : g \in G \mapsto \gamma_g \in \text{Aut}(\mathcal{F})$ of a field group G (the *gauge group of 1st kind*) on another separable C^* -algebra \mathcal{F} (the *field algebra*). Due to the univalence superselection rule (cf. [49]), when we deal with the CAR algebra $\text{CAR}(J)$, the gauge group is precisely \mathbb{Z}_2 and the observable algebra is the even part $\text{CAR}(J)_+$. For general theories which include Fermionic systems, the gauge group includes \mathbb{Z}_2 , and the even part of the field algebra is the invariant part under the action of the generator $\sigma \in \mathbb{Z}_2 \subset G$.

We say that the C^* -algebra \mathcal{A} is *graded asymptotically Abelian* w.r.t. α , if for each $A, B \in \mathcal{A}_\pm$,

$$\lim_{|x| \rightarrow +\infty} \{\alpha_x(A), B\}_\epsilon = 0,$$

In order to introduce the disorder, we consider a standard measure space (Ω, μ) based on a compact separable space Ω , and a Borel probability measure μ . The group \mathbb{Z}^d of the spatial translations is supposed to act on the probability space (Ω, μ) by measure preserving transformations $\{T_x\}_{x \in \mathbb{Z}^d}$.

A one parameter random group of automorphisms

$$(t, \omega) \in \mathbb{R} \times \Omega \mapsto \tau_t^\omega \in \text{Aut}(\mathcal{A})$$

is acting on \mathcal{A} . It is by definition a representation of \mathbb{R} for each fixed realization $\omega \in \Omega$ of the coupling constants. Furthermore, it is supposed to be jointly measurable in the σ -strong topology. As a consequence of the Banach–Kuratowski–Pettis Theorem (cf. [31], pag. 211), for each fixed value $\omega \in \Omega$, the one parameter group $t \in \mathbb{R} \mapsto \tau_t^\omega \in \text{Aut}(\mathcal{A})$ is automatically continuous in the σ -strong topology.

Consider, for $A \in \mathcal{A}$, the measurable function $f_{A,t}(\omega) := \tau_t^\omega(A)$. We get

$$\|f_{A,t}\|_{L^\infty(\Omega, \mu; \mathcal{A})} \equiv \text{esssup}_{\omega \in \Omega} \|\tau_t^\omega(A)\|_{\mathcal{A}} = \|A\|_{\mathcal{A}},$$

where the last equality follows as τ_t^ω is isometric. As in the concrete example $\mathcal{A} = \text{CAR}(\mathbb{Z}^d)$ in Section 2, we further assume that τ acts locally. Namely, if $A \in \mathcal{A}$, then the function $f_{A,t} \in L^\infty(\Omega, \mu; \mathcal{A})$ belongs to the C^* -subalgebra $\mathcal{A} \otimes L^\infty(\Omega, \mu)$.⁵

Finally, we assume the following commutation rule

$$(3.4) \quad \tau_t^{T_x \omega} \alpha_x = \alpha_x \tau_t^\omega, \quad x \in \mathbb{Z}^d, \omega \in \Omega, t \in \mathbb{R}.$$

By following the approach of encoding the disorder in a bigger algebra (cf. [4, 8, 9, 10, 23, 32]), the disordered system under consideration is described by

$$\mathfrak{A} := \mathcal{A} \otimes L^\infty(X, \nu).$$

Notice that, by identifying \mathfrak{A} with a closed subspace of $L^\infty(X, \nu; \mathcal{A})$, each element $A \in \mathfrak{A}$ is uniquely represented by a measurable, essentially bounded function $\omega \mapsto A(\omega)$ with values in \mathcal{A} . In addition, \mathfrak{A} contains copies $\mathcal{A} \otimes \mathbf{1}$ and $\mathbf{1} \otimes L^\infty(\Omega, \mu)$ of \mathcal{A} and $L^\infty(\Omega, \mu)$ respectively, denoted also by \mathcal{A} and $L^\infty(\Omega, \mu)$, by an abuse of notation.

⁵In the present paper, $\mathcal{A} \otimes L^\infty(\Omega, \mu)$ means the C^* -algebra obtained by completing the algebraic tensor product $\mathcal{A} \odot L^\infty(\Omega, \mu)$ under any C^* -cross norm, as any Abelian C^* -algebra is nuclear.

The group \mathbb{Z}^d of all the space translations acts naturally on the C^* -algebra \mathfrak{A} as

$$(3.5) \quad \mathfrak{a}_x(A)(\omega) := \alpha_x(A(T_{-x}\omega)), \quad A \in \mathfrak{A}, \omega \in \Omega, x \in \mathbb{Z}^d.$$

Further, as the time translations are supposed to act locally,

$$\mathfrak{t}_t(A)(\omega) := \tau_t^\omega(A(\omega)), \quad A \in \mathfrak{A}, \omega \in \Omega, t \in \mathbb{R}$$

is a well defined one parameter group of automorphisms of \mathfrak{A} , continuous in the σ -strong topology. In addition, put

$$\mathfrak{s} := \sigma \otimes \text{id}_{L^\infty(X, \nu)}.$$

Then the subspaces \mathfrak{A}_+ and \mathfrak{A}_- are defined as in (3.2).

On account of (3.4) and (3.5), it is straightforward to verify that $\{\mathfrak{a}_x\}_{x \in \mathbb{Z}^d}$ and $\{\mathfrak{t}_t\}_{t \in \mathbb{R}}$ define actions of \mathbb{Z}^d and \mathbb{R} respectively on \mathfrak{A} which are mutually commuting. Furthermore, by (3.1), $\mathfrak{a}_x \mathfrak{s} = \mathfrak{s} \mathfrak{a}_x$ for each $x \in \mathbb{Z}^d$. Concerning the parity of the time translations, we assume that $\mathfrak{t}_t \mathfrak{s} = \mathfrak{s} \mathfrak{t}_t$ as well. In the concrete cases under consideration, the parity of the time translations will follow by the fact that the time translations and the spatial translations are mutually commuting. Indeed, we have

Proposition 3.1. *Suppose $\mathcal{A} = \text{CAR}(\mathbb{Z}^d)$. Under all the previous assumptions except the parity for the time evolution, we get $\mathfrak{t}_t \mathfrak{s} = \mathfrak{s} \mathfrak{t}_t$.*

Proof. If $A, B \in \mathfrak{A}$ with $B(\omega) = B$ a constant field, we see that $\{\mathfrak{a}_x(A), B\}_\epsilon \rightarrow 0$ when $|x| \rightarrow +\infty$. By reasoning as in Lemma 8.2 of [6], we see that

$$(3.6) \quad \lim_{|x| \rightarrow +\infty} \|[\mathfrak{a}_x(A), B]\| = 0$$

for each $B \in \mathfrak{A}$, if and only if A is even. Indeed, let $A \in \mathfrak{A}$ and B a constant field made by the unitary $U = a_{x_0} + a_{x_0}^\dagger$ for any choice of $x_0 \in \mathbb{Z}^d$. We get

$$\begin{aligned} [\mathfrak{a}_x(A), B] &= [\mathfrak{a}_x(A_+), B] + [\mathfrak{a}_x(A_-), B] \\ &= [\mathfrak{a}_x(A_+), B] + \{\mathfrak{a}_x(A_-), B\} - 2\mathfrak{a}_x(A_-)B. \end{aligned}$$

If (3.6) holds true, then $\mathfrak{a}_x(A_-)B \rightarrow 0$ as the first two terms in the l.h.s. go to zero due to CAR. Thus $\|A_-\| = \|\mathfrak{a}_x(A_-)\| = \|\mathfrak{a}_x(A_-)B\| \rightarrow 0$, that is $A = A_+$. The converse statement follows from the graded asymptotic Abelianness of the CAR algebra $\text{CAR}(\mathbb{Z}^d)$ w.r.t. the spatial translations. The proposition now follows by applying the reasoning in the proof of proposition 8.1 of [6] to the time translations and spatial translations on the disorder algebra \mathfrak{A} . \square

4. STATES

Consider a state $\varphi \in \mathcal{S}(\mathfrak{A})$ which is invariant w.r.t. the spatial translation \mathfrak{a} . Let $(\mathcal{H}_\varphi, \pi_\varphi, U_x, \Phi)$ be the GNS covariant quadruple associated to φ .

Let $C, D \in \mathfrak{A}$ and $A, B \in \mathfrak{A}_\pm$. We say that the state φ is *graded asymptotically Abelian* w.r.t. \mathfrak{a} if

$$(4.1) \quad \lim_{|x| \rightarrow +\infty} \varphi(C\{\mathfrak{a}_x(A), B\}_\epsilon D) = 0,$$

where $\{\cdot, B\}_\epsilon$ is the graded commutator given in (3.3)

The state φ is *weakly clustering* w.r.t. \mathfrak{a} if

$$(4.2) \quad \lim_N \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi(A\mathfrak{a}_x(B)) = \varphi(A)\varphi(B),$$

Λ_N being the box with a vertex located at the origin, containing N^d points with positive coordinates.⁶

The state φ is \mathbb{Z}^d -Abelian if $E_\varphi \pi_\varphi(\mathfrak{A}) E_\varphi \subset \mathcal{B}(\mathcal{H}_\varphi)$ is a family of mutually commuting operators, E_φ being the selfadjoint projection onto the invariant vectors for the action of U_x . Furthermore, a state $\varphi \in \mathcal{S}(\mathfrak{A})$ is *even* if it is σ -invariant. Denote by $\mathcal{S}(\mathfrak{A})_+$ the set of all the even states.

We report the following result for the sake of completeness.

Proposition 4.1. *Suppose that $\varphi \in \mathcal{S}(\mathfrak{A})$ is a \mathfrak{a} -invariant, graded asymptotically Abelian state. Then $\varphi \in \mathcal{S}(\mathfrak{A})_+$ and it is \mathbb{Z}^d -Abelian. In addition, the following assertions are equivalent.*

- (i) φ is \mathfrak{a} -weakly clustering,
- (ii) φ is \mathfrak{a} -ergodic.

Proof. By reasoning as in Example 5.2.21 of [15], we conclude that φ is automatically even and \mathbb{Z}^d -Abelian. Concerning the last assertion, it is a well-known fact that (i) always implies (ii). The reverse implication follows as in Proposition 5.4.23 of [15], the last working also under the weaker condition (4.1). \square

In the present paper, the asymptotic Abelianess is always w.r.t. the spatial translations if it is not automatically specified.

⁶For continuous dynamical systems, one uses in (4.2) the natural modification M of the Cesaro mean given on bounded measurable functions, given by

$$M(f) := \lim_{D \rightarrow +\infty} \frac{1}{\text{vol}(\Lambda_D)} \int_{\Lambda_D} f(x) \, d^d x,$$

Λ_D being a box with edges of length D .

Let π be a representation of \mathfrak{A} . We easily get

$$\pi(L^\infty(\Omega, \mu)) \subset \mathfrak{Z}_\pi.$$

Suppose that π is normal when restricted to $L^\infty(\Omega, \mu)$. In such a situation, there exists an essentially unique measurable set $E \subset \Omega$, such that

$$\pi(L^\infty(\Omega, \mu)) \sim L^\infty(\Omega, \nu).$$

where ν is nothing but the absolutely continuous measure w.r.t. μ given, for each measurable set F , by

$$\nu(F) = \mu(F \cap E).$$

We have also

$$\pi(\mathfrak{A})'' = \pi(\mathcal{A} \otimes C(\Omega))'',$$

that is $\pi(\mathcal{A} \otimes C(\Omega))$ is a weakly dense separable C^* -subalgebra of $\pi(\mathfrak{A})''$.

We can consider the subcentral decomposition of the restriction of π to the separable C^* -subalgebra $\mathcal{A} \otimes C(\Omega)$, w.r.t. $\pi(L^\infty(\Omega, \mu)) \equiv \pi(C(\Omega))''$, see [52], Theorem IV 8.25. We obtain

$$(4.3) \quad \pi = \int_{\Omega}^{\oplus} \pi_{\omega} \mu(d\omega)$$

on

$$\mathcal{H}_{\pi} = \int_{\Omega}^{\oplus} \mathcal{H}_{\omega} \mu(d\omega).$$

The measurable field $\{\pi_{\omega}\}_{\omega \in \Omega}$ of representations of $\mathcal{A} \otimes C(\Omega)$ is uniquely determined by its restriction to \mathcal{A} .⁷ This follows from the fact that for each $A \in \mathcal{A}$ and $f \in L^\infty(\Omega, \mu)$, we have

$$(4.4) \quad \pi(A \otimes f) = \int_{\Omega}^{\oplus} f(\omega) \pi_{\omega}(A \otimes \mathbf{1}) d\mu(\omega).$$

Now by Lemma 8.4.1 of [17], we have

$$M := \pi(\mathfrak{A})'' = \int_{\Omega}^{\oplus} M_{\omega} \nu(d\omega),$$

where for almost all $\omega \in \Omega$,

$$M_{\omega} = \pi_{\omega}(\mathcal{A} \otimes L^\infty(\Omega, \mu))'' \equiv \pi_{\omega}(\mathcal{A} \otimes \mathbf{1})''.$$

As explained in [9, 23], in order to take into account the disorder, we deal with states $\varphi \in \mathcal{S}(\mathfrak{A})$ which are normal when restricted to $L^\infty(\Omega, \mu)$ (i.e. $\varphi(\mathbf{1} \otimes f) = \int f g_{\varphi} d\mu$ for a uniquely determined $g_{\varphi} \in$

⁷For $\omega \in E^c$, the complement of the support E of ν , π_{ω} will be the trivial representation on the trivial Hilbert space $\mathcal{H}_{\omega} \equiv \{0\}$.

$L^1(\Omega, \mu)$). Then there exists ([52], Proposition IV.8.34) a $*$ -weak measurable field $\{\varphi_\omega\}_{\omega \in \Omega}$ of positive forms on \mathcal{A} such that, for each $A \in \mathfrak{A}$,

$$(4.5) \quad \varphi(A) = \int_{\Omega} \varphi_\omega(A(\omega)) \mu(d\omega),$$

the function $\omega \mapsto A(\omega)$ being the representative of A in $L^\infty(\Omega, \mu; \mathcal{A})$.

Consider the GNS representation π_φ relative to φ . It is straightforward to check that, for almost all $\omega \in \Omega$, π_{φ_ω} is unitarily equivalent to the restriction of π_ω to $\mathcal{A} \otimes \mathbf{1} \sim \mathcal{A}$, where π_{φ_ω} is the GNS representation of φ_ω , and π_ω is the representation occurring in the decomposition of π_φ as given in (4.3).

Proposition 4.2. *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be such that $\varphi|_{L^\infty(\Omega, \mu)}$ is normal. If it is invariant w.r.t. \mathfrak{a} , then for almost all $\omega \in \Omega$, the form φ_ω in (4.5) is even.*

Proof. As φ is invariant w.r.t. the spatial translation, $\varphi \in \mathcal{S}(\mathfrak{A})_+$ (cf. Proposition 4.1). Then

$$\varphi(A \otimes f) = \int_{\Omega} f(\omega) \varphi_\omega(A) d\mu(\omega) = 0$$

for each $f \in L^\infty(\Omega, \mu)$ and $A \in \mathcal{A}_-$. Thus, for each $A \in \mathcal{A}_-$ there exists a measurable set $\Omega_A \subset \Omega$ of full measure such that $\varphi_\omega(A) = 0$ on Ω_A . As $\mathcal{A} \sim \mathcal{A}_+ \oplus \mathcal{A}_-$ as a Banach space, we can find a countable dense set $\mathcal{X} \subset \mathcal{A}_-$. Then for each $A \in \mathcal{X}$ we have $\varphi_\omega(A) = 0$, simultaneously on the measurable set $\Omega_0 := \bigcap_{A \in \mathcal{X}} \Omega_A \subset \Omega$ of full measure. Fix $A \in \mathcal{A}_-$ and choose sequences $\{A_n\} \subset \mathcal{X}$ converging to A . Then we have on Ω_0 ,

$$\varphi_\omega(A) = \varphi_\omega(\lim_n A_n) = \lim_n \varphi_\omega(A_n) = 0,$$

that is the positive forms φ_ω given in (4.5) are even almost surely. \square

We now show how the graded asymptotic abelianness of the states on \mathfrak{A} under consideration directly follows from that of \mathcal{A} .

Proposition 4.3. *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be such that $\varphi|_{L^\infty(\Omega, \mu)}$ is normal. If \mathcal{A} is graded asymptotically Abelian then for each $C \in \mathfrak{A}$ and $A, B \in \mathfrak{A}_\pm$ we have,*

$$\lim_{|x| \rightarrow +\infty} \varphi(C^* \{ \mathfrak{a}_x(A), B \}_\epsilon^* \{ \mathfrak{a}_x(A), B \}_\epsilon C) = 0.$$

In particular, if φ is invariant w.r.t. \mathfrak{a} , it is graded asymptotically Abelian.

Proof. Let $f, g, h \in L^\infty(\Omega, \mu)$, and $A, B \in \mathcal{A}_\pm$, $C \in \mathcal{A}$. We obtain by (3.5) and (4.5),

$$\begin{aligned} & \varphi((C \otimes h)^* \{\mathfrak{a}_x(A \otimes f), B \otimes g\}_\epsilon^* \{\mathfrak{a}_x(A \otimes f), B \otimes g\}_\epsilon C \otimes h) \\ &= \int_\Omega |f(T_{-x}\omega)g(\omega)h(\omega)|^2 \varphi_\omega(C^* \{\alpha_x(A), B\}_\epsilon^* \{\alpha_x(A), B\}_\epsilon C) d\mu(\omega) \\ &\leq (\|f\|_\infty \|g\|_\infty \|h\|_\infty \|C\|)^2 \|\{\alpha_x(A), B\}_\epsilon\|^2 \rightarrow 0, \end{aligned}$$

as \mathcal{A} is graded asymptotically Abelian w.r.t. the spatial translations. This means $\|\pi_\varphi(\{\alpha_x(A), B\}_\epsilon)\xi\| \rightarrow 0$ on the Hilbert space of the GNS representation of φ , which implies

$$\lim_{|x| \rightarrow +\infty} \varphi(C \{\mathfrak{a}_x(A), B\}_\epsilon D) = 0$$

for each $A, B \in \mathfrak{A}_\pm$ and $C, D \in \mathfrak{A}$ □

Next we recall the definition of the KMS boundary condition which is useful for the description of the temperature states of a quantum dynamical system, see e.g. [15].

A state ϕ on the C^* -algebra \mathfrak{B} satisfies the KMS boundary condition at inverse temperature β which we suppose to be always different from zero, w.r.t the group of automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$ if

- (i) $t \mapsto \phi(A\tau_t(B))$ is a continuous function for every $A, B \in \mathfrak{B}$,
- (ii) $\int \phi(A\tau_t(B))f(t)dt = \int \phi(\tau_t(B)A)f(t+i\beta)dt$ whenever $f \in \widehat{\mathcal{D}}$, \mathcal{D} being the space made of all infinitely often differentiable compactly supported functions in \mathbb{R} .

For the equivalent characterizations of the KMS boundary condition, the main results about KMS states, and finally the connections with Tomita theory of von Neumann algebras, see e. g. [14, 15, 47] and the references cited therein.

It is well known that the cyclic vector Ω_ϕ of the GNS representation π_ϕ is also separating for $\pi_\phi(\mathfrak{B})''$. Denote by σ^ϕ its modular group. According to the definition of KMS boundary condition, we have

$$(4.6) \quad \sigma_t^\phi \circ \pi_\phi = \pi_\phi \circ \tau_{-\beta t}.$$

We end the present section by listing some useful properties of states, which are normal (not necessarily KMS) when restricted to $L^\infty(\Omega, \mu)$, contained in Section 3 and 4 of [9].

Proposition 4.4. *Let φ be \mathfrak{t} -KMS state on \mathfrak{A} at inverse temperature β which is normal when restricted to $L^\infty(\Omega, \mu)$.*

Then, for almost all $\omega \in \Omega$, the forms φ_ω given in (4.5) are τ^ω -KMS at the same inverse temperature β .

Proof. The proof is along the same lines as those of Proposition 4.2, which is reported in Section 3 of [9]. \square

Theorem 4.5. *Let φ be an invariant state on \mathfrak{A} which is normal if restricted to $L^\infty(\Omega, \mu)$. Consider the decomposition appearing in (4.5). Then*

- (i) $\varphi_\omega \circ \alpha_x = \varphi_{T_{-x}\omega}$ for all $x \in \mathbb{Z}_d$,
- (ii) $\mathfrak{Z}_{\pi_{\varphi_\omega}} \cong \mathfrak{Z}_{\pi_{\varphi_{T_x\omega}}}$ for all $x \in \mathbb{Z}_d$,

where the above equalities, as well as the unitary equivalence, are satisfied almost everywhere.

In addition, if the action T_x on Ω is ergodic, then $\varphi(\mathbf{1}) = 1$ almost surely.

Consider for $\alpha \in \{\infty\} \cup \{1, 2, \dots\} \cup \{\lambda_\infty, \lambda_0, \lambda_1, \dots\}$, the Abelian von Neumann algebras $L^\infty(E_\alpha, \nu_\alpha)$ defined as follows. For $\alpha \in \{\infty\} \cup \{1, 2, \dots\}$, (E_n, ν_n) is the countable set $E_n = \mathbf{n}$ of cardinality n , equipped with the counting measure ν_n (the symbol ∞ corresponds to the denumerable cardinality). For $\alpha = \lambda_n$, $(E_{\lambda_n}, \nu_{\lambda_n})$ is the disjoint union $[0, 1] \cup \mathbf{n}$ equipped with the measure ν_{λ_n} made of the Lebesgue measure λ on $[0, 1]$, and the counting measure on \mathbf{n} (the value $n = 0$ corresponds to $L^\infty([0, 1], \lambda)$).

As a corollary to Theorem 4.5, we have

Corollary 4.6. *Let φ be as in Theorem 4.5, and suppose that T_x acts ergodically on Ω . Then there exists a unique $\alpha \in \{\infty\} \cup \{1, 2, \dots\} \cup \{\lambda_\infty, \lambda_0, \lambda_1, \dots\}$ such that*

$$\mathfrak{Z}_{\pi_{\varphi_\omega}} \sim L^\infty(E_\alpha, \nu_\alpha)$$

almost surely.

We cannot conclude that $\mathfrak{Z}_{\pi_{\varphi_\omega}}$ is almost surely of a unique multiplicity class.⁸ However, for locally normal invariant KMS states we have that $\mathfrak{Z}_{\pi_{\varphi_\omega}}$ is almost surely of infinite multiplicity, see below.

Definition 4.7. *For the \mathbb{Z}_2 -graded models considered in the present paper, we denote by $\mathcal{S}_N(\mathfrak{A}) \subset \mathcal{S}(\mathfrak{A})_+$ the subset of the even states φ such that $\varphi|_{L^\infty(\Omega, \mu)}$ is normal. The set $\mathcal{S}_{NI}(\mathfrak{A}) \subset \mathcal{S}_N(\mathfrak{A})$ consists of those states $\varphi \in \mathcal{S}_N(\mathfrak{A})$ which are in addition \mathfrak{a} -invariant.*

In the classical setting, the measurable field $\{\varphi_\omega\}$ arising from the direct integral decomposition of a temperature state $\varphi \in \mathcal{S}_{NI}(\mathfrak{A})$ is called an *Aizenman–Wehr metastate*, see [2]. In the quantum case,

⁸Notice that there are uncountable many Abelian von Neumann algebras acting on separable Hilbert spaces, up to unitary equivalence.

the counterparts of the Aizenman–Wehr metatstates were naturally considered early in [32].

5. SPECTRAL PROPERTIES OF \mathbb{Z}_2 –GRADED ASYMPTOTICALLY ABELIAN DYNAMICAL SYSTEMS

The present section is devoted to prove some useful results concerning the spectral properties of dynamical systems which are \mathbb{Z}_2 –graded asymptotically Abelian, the last being the natural setting for theories including Fermi particles. The results proved below and in the next section have a self contained interest as they provide the generalization to the \mathbb{Z}_2 –graded dynamical systems of the pivotal results of [3, 28, 46], and those reviewed in [35] for the natural applications to the investigation of the structure of the local algebras in Quantum Field Theory. For the definition and the main properties of the Borchers, Connes and Arveson spectra $\Gamma_B(\alpha)$, $\Gamma(\alpha)$, $\text{sp}(\alpha)$ of an action α , and then the Borchers and Connes invariants $\Gamma_B(M)$, $\Gamma(M)$ of a von Neumann algebra M , respectively, the reader is referred to the original papers [13, 16] and the books [44, 50].

One of the main objects of interest in the investigation of the spectral properties of non commutative dynamical systems is the Arveson spectrum and its connection with the spectrum of the group of unitaries implementing the dynamics in the covariant GNS representation, see e.g. [44]. It was shown that for the dynamical systems based on random interactions treated in [8, 9] the Arveson spectrum is almost surely independent of the disorder.⁹ In addition, for most of the KMS states considered in [8], the spectrum of the associated modular group was also found to be independent of the disorder. The proofs of such results depend mainly on the general properties assumed for our disordered model and not so much on the local structure of the C^* –algebra \mathcal{A} . Therefore the proofs of these results which appear as Theorem 5.3 and Proposition 5.5 in [8] can be reproduced *mutatis mutandis* for the situation under consideration in the present paper. To this end, we suppose that \mathcal{A} is a separable unital C^* –algebra. Let τ_t^ω be jointly measurable in t, ω . Assume the commutation rule (3.4) and the ergodicity of the action T_x of the spatial translations on the sample space (Ω, μ) . Put $\mathfrak{A} := \mathcal{A} \otimes L^\infty(\Omega, \mu)$ and choose a KMS state $\varphi \in \mathcal{S}_{NI}(\mathfrak{A})$ at inverse temperature $\beta \neq 0$. Consider the forms φ_ω in (4.5), which thanks to Propositions 4.2 and 4.4, are even and satisfy the KMS condition at the same inverse temperature β almost surely.

⁹Compare with the analogous result ([34], Théorème III.1) concerning the spectrum of a one dimensional random discretized Schrödinger operator.

Theorem 5.1. *Under the above assumptions, there exists a measurable set $F \subset \Omega$ of full measure, and a closed set $\Sigma \subset \mathbb{R}$ such that $\omega \in F$ implies $\text{sp}(\tau^\omega) = \Sigma$.*

In addition, if \mathcal{A} is simple, then

$$\beta \text{sp}(\tau^\omega) = -\sigma(\ln \Delta_{\varphi_\omega})$$

almost surely, where Δ_{φ_ω} is the modular operator associated to φ_ω .

Proof. We get by [44], Proposition 8.1.9,

$$\text{sp}(\tau^\omega) = \bigcap_{f \in L^1(\mathbb{R})} \{s \in \mathbb{R} \mid |\hat{f}(s)| \leq \|\tau_f^\omega\|\}$$

where “ $\hat{}$ ” stands for (inverse) Fourier transform, and

$$\tau_f^\omega(A) := \int_{-\infty}^{+\infty} f(t) \tau_t^\omega(A) \, dt,$$

the integral being understood in the Bochner sense. By a standard density argument, we can reduce the situation to a dense set $\{f_k\}_{k \in \mathbb{N}} \subset L^1(\mathbb{R})$. Define $\Gamma_k(\omega) := \|\tau_{f_k}^\omega\|$. It was shown in [8] that the functions Γ_k are measurable and invariant. By ergodicity, they are constant almost everywhere. Let $\{N_k\}_{k \in \mathbb{N}}$ be null subsets of Ω such that, for each $k \in \mathbb{N}$ and $\omega \in N_k^c$,

$$\Gamma_k(\omega) = \|\Gamma_k\|_\infty.$$

Consider $F := (\bigcup_{k \in \mathbb{N}} N_k)^c$, and take $\Sigma := \text{sp}(\tau^{\omega_0})$, where ω_0 is any element of F . As an immediate consequence of this, we have that F is a measurable set of full measure, and $\omega \in F$ implies $\text{sp}(\tau^\omega) = \Sigma$.

Consider the GNS covariant representation $(\mathcal{H}_\omega, \pi_\omega, U_\omega, \Phi_\omega)$ of φ_ω . Thanks to the facts that, on a measurable set $F \subset \Omega$ of full measure, Φ_ω is a standard vector for $\pi_\omega(\mathcal{A})''$ and π_ω is faithful as \mathcal{A} is simple, we get for $f \in L^1(\mathbb{R})$, $U_\omega(f) := \int f(t) U_\omega(t) \, dt = 0$ if and only if $\tau_f^\omega \equiv \int f(t) \tau_t^\omega \, dt = 0$. For $\omega \in F$, this leads to $\text{sp}(\tau^\omega) = -\frac{1}{\beta} \sigma(\ln \Delta_{\varphi_\omega})$ by (4.6).¹⁰ \square

Now we generalize a standard result (cf. [3, 28, 46]) on the spectrum of the modular action of asymptotically Abelian systems to the \mathbb{Z}_2 -graded case. The proof in the graded case will be more involved than that for the asymptotically Abelian situation.

¹⁰Notice that if \mathcal{A} is not simple, we merely have $\text{sp}(\tilde{\tau}^\omega) = -\frac{1}{\beta} \sigma(\ln \Delta_{\varphi_\omega})$ where $\tilde{\tau}_t^\omega := \text{ad } U_\omega(t)$ acting on $\pi_{\varphi_\omega}(\mathcal{A})''$. It can be showed as before, that it is independent on the disorder.

Theorem 5.2. *Let (M, G, τ) , (M, H, α) be W^* -dynamical systems based on the \mathbb{Z}_2 -graded W^* -algebra M , with G locally compact and Abelian. Suppose that the actions τ and α are even, commute each other, and leave invariant the faithful normal state φ .*

If for an invariant mean m_H on H , and for each $A \in M_\pm$, $B \in Z(M^\tau)_\pm$,

$$(5.1) \quad m_H \{ \varphi (\{ \alpha_h(A), B \}_\epsilon^* \{ \alpha_h(A), B \}_\epsilon) \} = 0,$$

*then $\Gamma_B(\tau) = \text{sp}(\tau)$.*¹¹

Proof. Fix $E \in Z(M^\tau)$ with central support (in M) $c(E) = \mathbf{1}$. We notice that $\sigma(E) \in Z(M^\tau)$ as τ is even. In addition, $c(\sigma(E)) = \mathbf{1}$ too. Let $p \in \text{sp}(\tau)$ and fix a closed neighborhood V of p . Then there exists a nonzero element $A \in M(\tau, V)$, the last being the spectral subspace associated with the closed subset $V \subset \hat{G}$ (cf. [44]). If $M(\tau, V) \cap M_+ \neq \{0\}$, we argue as in Theorem 2 of [28], that there exists $h \in H$ such that $E\alpha_h(A)E \neq 0$ where $A \in M(\tau, V) \cap M_+$. In this case $E\alpha_h(A)E \in M_E(\tau^E, V)$, where τ^E is the restricted action of τ on the reduced algebra M_E . If for some closed neighborhood V of p , $M(\tau, V) \subset M_-$, we proceed as follows. Namely, fix a nonzero element $A \in M(\tau, V)$ and suppose that $E\alpha_h(A)E \neq 0$ for some $h \in H$. Then we conclude that $E\alpha_h(A)E \in M_E(\tau^E, V)$, where τ^E is the restricted action of τ on the reduced algebra M_E . If on the other hand, $E\alpha_h(A)E = 0$ for each $h \in H$, then we get,

$$(5.2) \quad \begin{aligned} E\sigma(E)\alpha_h(A^*A)\sigma(E)E &= \sigma(E)E\alpha_h(A^*)E\alpha_h(A)E\sigma(E) \\ &+ E\sigma(E)\alpha_h(A^*)\{\alpha_h(A), \sigma(E)\}_\epsilon E = E\sigma(E)\alpha_h(A^*)\{\alpha_h(A), \sigma(E)\}_\epsilon E. \end{aligned}$$

Let now $\mathcal{E}_H : M \rightarrow M^\alpha$ (resp. $\mathcal{E}_G : M \rightarrow M^\tau$) be the normal faithful conditional expectation onto the fixed point subalgebra M^α (resp. M^τ) leaving invariant the state φ , which exists by the Kovacs–Szűcs Theorem (see e.g. [14], Proposition 4.3.8). We have by Cauchy–Schwarz Inequality, Holder Inequality, (5.1) and (5.2),

$$\begin{aligned} \varphi(E\sigma(E)\mathcal{E}_H(A^*A)\sigma(E)E) &= m_H \{ \varphi(E\sigma(E)\alpha_h(A^*A)\sigma(E)E) \} \\ &= m_H \{ \varphi(E\sigma(E)\alpha_h(A^*)\{\alpha_h(A), \sigma(E)\}_\epsilon E) \} \\ &\leq \|A\| m_H \{ \varphi(\{\alpha_h(A), \sigma(E)\}_\epsilon^* \{\alpha_h(A), \sigma(E)\}_\epsilon) \}^{1/2} = 0. \end{aligned}$$

This implies that $E\sigma(E)\mathcal{E}_H(A^*A)\sigma(E)E = 0$ as φ is faithful. Now,

$$\mathcal{E}_G(\sigma(E)\mathcal{E}_H(A^*A)\sigma(E))E = \mathcal{E}_G(E\sigma(E)\mathcal{E}_H(A^*A)\sigma(E)E) = 0$$

¹¹For the definition of invariant means of a (semi)group, we refer the reader to Section 17 of [29].

as $E \in Z(M^\tau)$. In addition, $\mathcal{E}_G(\sigma(E)\mathcal{E}_H(A^*A)\sigma(E)) = 0$ as $c(E) = \mathbf{1}$, which implies $\sigma(E)\mathcal{E}_H(A^*A)\sigma(E) = 0$ as \mathcal{E}_G is faithful. By repeating the same argument for $\sigma(E)\mathcal{E}_H(A^*A)\sigma(E)$ we get $\mathcal{E}_H(A^*A) = 0$, which implies the contradiction $A = 0$ as \mathcal{E}_H is faithful as well. Thus, either when $M(\tau, V) \cap M_+ \neq \{0\}$ or $M(\tau, V) \subset M_-$, if $p \in \text{sp}(\tau)$ and V is any closed neighborhood V of p , there always exists $A \in M$ such that $E\alpha_h(A)E \neq 0$ and $E\alpha_h(A)E \in M_E(\tau^E, V)$. This means that $p \in \text{sp}(\tau^E)$ as well. As by Proposition 1 of [28],

$$\Gamma_B(\tau) = \bigcap_{\{E \in Z(M^\tau) | c(E) = \mathbf{1}\}} \text{sp}(\tau^E),$$

this leads to the assertion. \square

6. THE TYPE OF VON NEUMANN ALGEBRAS ASSOCIATED TO GRADED ASYMPTOTICALLY ABELIAN DYNAMICAL SYSTEMS AND FERMIONIC DISORDERED MODELS

The present section is devoted to investigate the type of von Neumann algebras associated to \mathbb{Z}_2 -graded asymptotically Abelian dynamical systems. The natural application will concern the von Neumann algebras generated by temperature states of disordered Fermionic models.

We start by proving that a von Neumann algebra with a nontrivial infinite semifinite summand cannot carry an action which is graded asymptotically Abelian w.r.t. the strong operator topology. The proof for the cases which include Fermionic systems is more involved than the original one in [18, 35]. As usual M will be a \mathbb{Z}_2 -graded von Neumann algebra whose grading is generated by an automorphism $\sigma \in \text{Aut}(M)$ with $\sigma^2 = \text{id}$.

Lemma 6.1. *Let M be a \mathbb{Z}_2 -graded semifinite von Neumann algebra with a normal semifinite faithful trace τ . Then there exists a selfadjoint projection $E \in M_+$ with $0 < \tau(E) < +\infty$.*

Proof. Choose a selfadjoint projection $F \in M$ with $0 < \tau(F) < +\infty$, and consider the splitting $F = F_+ + F_-$ of F into even and odd parts. As F is a selfadjoint projection, we get $F_+ = F_+^*F_+ + F_-^*F_-$. This leads to $0 < \tau(F_+) < +\infty$, otherwise $\tau(F_+^*F_+) + \tau(F_-^*F_-) = 0$ which implies $F_+ = F_- = 0$. Let $F_+ = \int \lambda dE(\lambda)$ be the resolution of the identity of F_+ . By considering first the approximation of continuous functions by polynomials in the uniform topology, and then the pointwise approximation of Borel functions with uniformly bounded continuous ones, we see that the projections $E(\lambda)$ are even. In addition, the increasing function $\lambda \mapsto \tau(E(\lambda))$ is the cumulative function associated with a

Borel measure on the interval $[0, 1]$ such that

$$\tau(F_+) = \int \lambda \, d\tau(E(\lambda)) .$$

Then there exists $\lambda_0 \in [0, 1]$ such that $0 < \tau(E(\lambda_0)) < +\infty$. The projection we are looking for is $E := E(\lambda_0) \in M_+$. \square

Consider on \mathbb{N} a mean m concentrated at the infinity of \mathbb{N} .¹²

Theorem 6.2. *Let M be a \mathbb{Z}_2 -graded von Neumann algebra. If for a mean m concentrated at the infinity of \mathbb{N} , and for a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset \text{Aut}(M)$ of even automorphisms,*

$$(6.1) \quad m \{ \varphi([\alpha_n(A), B]^*[\alpha_n(A), B]) \} = 0 ,$$

for each $A \in M$, $B \in M_+$ and $\varphi \in \mathcal{S}(M) \cap M_$, then the properly infinite semifinite summand in M is trivial.*

Proof. After using Lemma 6.1, the proof proceeds along the same lines as those of the analogous results in [18, 35]. Indeed, let F, G, H be the central projections of M corresponding to the finite, infinite semifinite, and purely infinite part of M , respectively. We have that all of them are invariant under the action of σ and the α_n , otherwise they would not be maximal. Thus, we can suppose that M is infinite semifinite. Choose a normal semifinite faithful trace τ (cf. [52], Theorem V.2.15) and a even selfadjoint projection $E \in M_+$ with $\tau(E) = 1$, which exists by Lemma 6.1. Consider on M the state

$$\varphi(A) := \tau(EAE), \quad A \in M .$$

Define $\psi \in \mathcal{S}(M)$ as

$$\psi(A) := m \{ \varphi \circ \alpha_n(A) \} , \quad A \in M .$$

By taking into account (6.1), we have by the Cauchy–Schwarz Inequality (cf. [52], Proposition I.9.5), and Holder inequality,

$$(6.2) \quad \begin{aligned} & |m \{ \varphi(\alpha_n(A)[\alpha_n(B), E]) \}| \\ & \leq \|A\| m \{ \varphi([\alpha_n(B), E]^*[\alpha_n(B), E]) \}^{1/2} = 0 . \end{aligned}$$

¹²The mean m concentrated at infinity is uniquely determined by a state on the Corona Algebra. Namely, $m \in \mathcal{S}(B(\mathbb{N})/B_0(\mathbb{N}))$, where $B(\mathbb{N})$ is the C^* -algebra of all the bounded functions on \mathbb{N} , and $B_0(\mathbb{N})$ is made of those bounded functions vanishing at infinity.

Thanks to (6.2), we compute

$$\begin{aligned}
\psi(AB) &= m \{ \tau(E\alpha_n(A)\alpha_n(B)E) \} = m \{ \varphi(\alpha_n(A)[\alpha_n(B), E]) \} \\
&+ m \{ \tau(E\alpha_n(A)E\alpha_n(B)E) \} = m \{ \tau(E\alpha_n(B)E\alpha_n(A)E) \} \\
&= m \{ \varphi(\alpha_n(B)[\alpha_n(A), E]) \} + m \{ \tau(E\alpha_n(B)E\alpha_n(A)E) \} \\
&= m \{ \tau(E\alpha_n(B)\alpha_n(A)E) \} = \psi(BA).
\end{aligned}$$

Namely, ψ is a (possibly non normal) tracial state on M which is a contradiction.¹³ \square

Consider a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of even $*$ -automorphisms of a \mathbb{Z}^2 -graded C^* -algebra \mathfrak{B} , together with an even state φ on \mathfrak{B} which is invariant for $\{\alpha_n\}$. Let $\{U_n\}$, V be the covariant implementation of $\{\alpha_n\}$, and of the grading σ relative to the GNS triplet $(\pi_\varphi, \mathcal{H}_\varphi, \Psi_\varphi)$ corresponding to φ , respectively. Denote by $\tilde{\alpha}_n := \text{ad}U_n$, $\tilde{\sigma} := \text{ad}V$ the corresponding automorphisms on $M := \pi_\varphi(\mathfrak{B})''$.

Lemma 6.3. *Suppose that Ψ_φ is separating for $\pi_\varphi(\mathfrak{B})''$.¹⁴ Then*

$$\lim_n \varphi(\{\alpha_n(A), B\}_\epsilon^* \{\alpha_n(A), B\}_\epsilon) = 0$$

for every $A, B \in \mathfrak{B}$ implies

$$\lim_n \{\tilde{\alpha}_n(X), Y\}_\epsilon \xi$$

for every $X, Y \in \pi_\varphi(\mathfrak{B})''$ and $\xi \in \mathcal{H}_\varphi$.

Proof. As $\Psi \equiv \Psi_\varphi$ is cyclic for M' , it is enough to show that $n \rightarrow \infty$ implies $\{\tilde{\alpha}_n(X), Y\}_\epsilon \Psi \rightarrow 0$.

Let $\varepsilon > 0$ and $X, Y \in M_1$ be fixed. Then there exist $X', Y' \in M' \setminus \{0\}$ such that

$$\|(X - X')\Psi\| < \varepsilon, \quad \|(Y - Y')\Psi\| < \varepsilon.$$

Let $X, Y \in M_\pm$, we can find $A, B \in \mathfrak{B}_\pm$ with $\|\pi_\varphi(A)\| \leq 1$, $\|\pi_\varphi(B)\| \leq 1$, such that

$$\begin{aligned}
\|(X - \pi_\varphi(A))\Psi\| &< (1 \wedge 1/\|Y'\|)\varepsilon, \\
\|(Y - \pi_\varphi(B))\Psi\| &< (1 \wedge 1/\|X'\|)\varepsilon.
\end{aligned}$$

¹³Choose selfadjoint mutually orthogonal projections $E_j \in M$, $j = 1, 2$ equivalent to the identity $\mathbf{1}$ such that $E_1 + E_2 = \mathbf{1}$, which always exist as M is properly infinite. Then $1 = \psi(\mathbf{1}) = \psi(E_1 + E_2) = \psi(E_1) + \psi(E_2) = \psi(\mathbf{1}) + \psi(\mathbf{1}) = 2$.

¹⁴Such a condition is equivalent to the fact that the state $(\cdot \Psi_\varphi, \Psi_\varphi)$ is a KMS state (at inverse temperature 1) on $\pi_\varphi(\mathfrak{B})''$ w.r.t. the modular automorphism group constructed from Ψ_φ . In addition, the previous conditions are also equivalent to the fact that the support $s(\varphi) \in \mathfrak{B}^{**}$ of the state φ in the bidual \mathfrak{B}^{**} is central, see e.g. [48], Section 10.17.

Indeed, for simplicity let $X \in M_-$ (the situation $X \in M_+$ follows analogously). Let P be the projection of M onto M_- . By the Kaplansky Density Theorem, there exists $C \in \mathfrak{B}$ such that

$$\|(X - \pi_\varphi(C))\Psi\| < (1 \wedge 1/\|Y'\|)\varepsilon.$$

Take $B := \frac{C - \sigma(C)}{2}$. By our assumptions, $X = P(X)$ and $\pi_\varphi(B) = P(\pi_\varphi(C))$. By taking into account the last, we get

$$\begin{aligned} \|(X - \pi_\varphi(B))\Psi\| &= \|(P(X - \pi_\varphi(B)))\Psi\| \\ &= \frac{1}{2}\|(X - \pi_\varphi(C))\Psi - V(X - \pi_\varphi(C))\Psi\| \\ &\leq \|(X - \pi_\varphi(C))\Psi\| < (1 \wedge 1/\|Y'\|)\varepsilon. \end{aligned}$$

We treat the situation $X, Y \in M_-$, the other cases being analogous.

$$\begin{aligned} \|(\{\tilde{\alpha}_n(X), Y\} - \pi_\varphi(\{\tilde{\alpha}_n(A), B\}))\Psi\| &\leq \|(\tilde{\alpha}_n(X)Y - \tilde{\alpha}_n(\pi_\varphi(A))\pi_\varphi(B))\Psi\| \\ &\quad + \|(Y\tilde{\alpha}_n(X) - \pi_\varphi(B)\tilde{\alpha}_n(\pi_\varphi(A)))\Psi\|. \end{aligned}$$

As both the terms of the r.h.s. of the above inequality is estimated in the same way, we consider only the first one. We get

$$\begin{aligned} \|(X\tilde{\alpha}_n(Y) - \pi_\varphi(A)\tilde{\alpha}_n(\pi_\varphi(B)))\Psi\| &\leq \|\tilde{\alpha}_n(X - \pi_\varphi(A))(Y - Y')\Psi\| \\ &\quad + \|Y'U_n(X - \pi_\varphi(A))\Psi\| + \|\pi_\varphi(\alpha_n(A))(Y - \pi_\varphi(B))\Psi\| \\ &< 2\varepsilon + \|Y'\|(1 \wedge 1/\|Y'\|)\varepsilon + (1 \wedge 1/\|X'\|)\varepsilon \leq 4\varepsilon \end{aligned}$$

which leads to the assertion. \square

As a direct consequence, we have the following result describing the structure of the von Neumann algebras generated by GNS representations associated with a \mathbb{Z}_2 -graded asymptotically Abelian state such that its support in the bidual is central. Such a result can be applied immediately to the model under consideration, and yet wider applications are possible.

Theorem 6.4. *Let $(\mathfrak{B}, \alpha, \varphi)$ be a C^* -dynamical system, with \mathfrak{B} a \mathbb{Z}_2 -graded C^* -algebra, α an even action of \mathbb{Z}^d , and finally φ an even state which is invariant under the action of α . Suppose that \mathcal{H}_φ is a separable Hilbert space and Ψ_φ is separating for $\pi_\varphi(\mathfrak{B})''$, $(\pi_\varphi, \mathcal{H}_\varphi, \Psi_\varphi)$ being the GNS triplet relative to φ . If*

$$\lim_{|x| \rightarrow +\infty} \varphi(\{\alpha_x(A), B\}_\epsilon^* \{\alpha_x(A), B\}_\epsilon) = 0,$$

then $\pi_\varphi(\mathfrak{B})''$ does not contain type I_∞ , II_∞ and III_0 components.

Proof. Let $\tilde{\varphi} := \langle \cdot \Psi_\varphi, \Psi_\varphi \rangle$. We start by noticing that $\Gamma_B(M) = \Gamma_B(\sigma^{\tilde{\varphi}})$ (cf. [28], Proposition 1) as the last does not depend on the faithful state $\tilde{\varphi}$ on M . By Proposition C1 of [9], $\text{Exp}(\Gamma_B(M)) = S(M) \setminus \{0\}$, $S(M)$ being Connes S -invariant (cf. [16]). Finally, $\text{sp}(\sigma^{\tilde{\varphi}}) = \text{ln } \sigma(\Delta_{\tilde{\varphi}})$.

Let E be the central projection corresponding to the type III_0 component of $\pi_\varphi(\mathfrak{B})'' \equiv M$ which is well-defined as M is acting on a separable Hilbert space, see [51]. Assume $E > 0$. As for $x \in \mathbb{Z}^d$, $\tilde{\alpha}_x(E) = E$, and $\tilde{\sigma}(E) = E$, we can suppose that $E = \mathbf{1}$, that is M is itself of type III_0 . As M is supposed of type III_0 , we get $S(M) = \{0, 1\}$. By considering the Cesaro mean as that described in (4.2), we obtain by Lemma 6.3 and Theorem 5.2, $\Gamma_B(M) = \text{sp}(\sigma^{\tilde{\varphi}})$. It readily follows from these results that,

$$\begin{aligned} \sigma(\Delta_{\tilde{\varphi}}) \setminus \{0\} &= \text{Exp}(\text{sp}(\sigma^{\tilde{\varphi}})) = \text{Exp}(\Gamma_B(\sigma^{\tilde{\varphi}})) \\ &= \text{Exp}(\Gamma_B(M)) = S(M) \setminus \{0\} = \{0, 1\} \setminus \{0\}. \end{aligned}$$

This means that $\sigma(\Delta_{\tilde{\varphi}}) = \{1\}$ as 0 cannot be an isolated point of the spectrum. Thus we have arrived at the contradiction that $\tilde{\varphi}$ is a trace. Hence, M cannot contain the type III_0 component. The proof follows as the infinite semifinite part is avoided by the application of Theorem 6.2, taking into account Lemma 6.3. \square

As a direct consequence of the previous results on the spectral properties, we show that the temperature states of the disordered systems under consideration can generate only type III von Neumann algebras, except the type III_0 .

Lemma 6.5. *Let \mathfrak{B} an infinite dimensional simple separable C^* -algebra together with its representation π . Then $\pi(\mathfrak{B})''$ does not contain the type I_{fin} component.*

Proof. Consider the central projection E_n relative to the I_n component, $n \in \mathbb{N}$, of $\pi(\mathfrak{B})''$, which exists by [43]. By considering the representation $\pi_n := \pi(\cdot)E_n$, we can assume that π itself contains only the type I_n component. Consider the direct integral decomposition $\pi = \int \pi_x d\nu$ of π w.r.t. the center of $\pi(\mathfrak{B})''$. We get that $\pi_x(\mathfrak{B})''$ is isomorphic to the full matrix algebra $\mathbb{M}_n(\mathbb{C})$, ν -almost surely. But this is impossible as π_x is faithful almost surely as \mathfrak{B} is simple, and then $\pi_x(\mathfrak{B})''$ cannot be $\mathbb{M}_n(\mathbb{C})$ as \mathfrak{B} is infinite dimensional. The proof follows as n is arbitrary and $E_{\text{fin}} = \bigoplus_{n \in \mathbb{N}} E_n$, E_{fin} being the central projection corresponding to the finite component of $\pi(\mathfrak{B})''$. \square

Theorem 6.6. *Let $\mathfrak{A} = \mathcal{A} \otimes L^\infty(\Omega, \mu)$, and $\varphi \in \mathcal{S}_{NI}(\mathfrak{A})$ be a KMS state at inverse temperature $\beta \neq 0$ w.r.t. the time evolution which is supposed to be nontrivial. Suppose that \mathcal{A} is separable, simple and*

graded asymptotically Abelian. Then only type III_λ factors, $\lambda \in (0, 1]$, can appear in its central decomposition.

If in addition, $\mathfrak{Z}_{\pi_\varphi} \sim L^\infty(\Omega, \mu)$, then there exists a unique $\lambda \in (0, 1]$ such that $\pi_{\varphi_\omega}(\mathcal{A})''$ are type III_λ factors almost surely.

Proof. Let $\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ be the direct integral decomposition of π as explained in Section 4. By taking into account (4.4), π_ω is indeed a representation of \mathcal{A} , and $\pi_\omega(\mathfrak{A})'' = \pi_\omega(\mathcal{A})''$ almost surely. By Lemma 6.5, we conclude that $\pi_\omega(\mathfrak{A})''$ cannot contain the type I_{fin} component, almost surely. This means that $\pi(\mathfrak{A})''$ does not contain the type I_{fin} component. By Proposition 4.3 and Theorem 6.4, the type I_∞ , II_∞ and III_0 are also absent.

Concerning the type II_1 component, let $E \in \mathfrak{Z}_{\pi_\varphi}$ be the corresponding central projection which we assume to be non zero. By proposition 3.1 of [9], the state

$$\varphi_E := \frac{\langle \pi_\varphi(\cdot) E \Phi, \Phi \rangle}{\langle E \Phi, \Phi \rangle}$$

is a KMS state which is normal w.r.t. φ . This means that $\varphi_E \upharpoonright_{L^\infty(\Omega, \mu)}$ is normal. In addition, $\varphi_E \in \mathcal{S}_{NI}(\mathfrak{A})$ as $VEV^* = E$ for each $V \in \mathcal{N}(\pi_{\varphi_\omega}(\mathfrak{A})'')$, $\mathcal{N}(\pi_\omega(\mathfrak{A})'')$ being the normalizer of $\pi_\varphi(\mathfrak{A})''$ in $\mathcal{B}(\mathcal{H}_\varphi)$. Thus, we assume without loss of generality that $\pi_\varphi(\mathfrak{A})''$ is a type II_1 von Neumann algebra. Denote as usual $\tilde{\tau}_t^\omega := \text{ad } U_\omega(t)$ and $\tilde{\mathfrak{t}}_t := \text{ad } U(t)$ on $\pi_{\varphi_\omega}(\mathfrak{A})''$ and $\pi_\varphi(\mathfrak{A})''$, respectively. By applying Proposition C2 of [9], theorems 5.1 and 5.2, we get

$$\text{sp}(\mathfrak{t}) = \text{sp}(\tau^\omega) = \text{sp}(\tilde{\tau}^\omega) = \text{sp}(\tilde{\mathfrak{t}}) = \Gamma_B(\tilde{\mathfrak{t}}) = \Gamma_B(\pi_\varphi(\mathfrak{A})'') = 0,$$

where the first three equalities hold true almost surely. But this is a contradiction as \mathfrak{t} is supposed to be non trivial, see e.g. [50], propositions 3.2.8 and 3.2.9.

If $\mathfrak{Z}_{\pi_\varphi} \sim L^\infty(\Omega, \mu)$, then $M_\omega \equiv \pi_{\varphi_\omega}(\mathcal{A})''$ are factors almost surely. This implies $\Gamma_B(M) = \Gamma(M_\omega)$ almost surely, where Γ is Connes Γ -invariant ([16]). The theorem follows from the previous part. \square

7. A CONCRETE DISORDERED FERMIONIC MODEL

We apply the previous results which are quite general in nature to a pivotal model. In fact, this model can be viewed as a disorderd spinless Hubbard model, provided the common distribution of the coupling constants is one-sides. The most interesting situation will be when such a common distribution is two-sides. The associated nearest neighbor random Hamiltonian of this Fermionic system we have in mind has the

form

$$(7.1) \quad H = \sum_{\{(x,y) \in \mathbb{Z}^d \mid |x-y|=1\}} \left(J_{xy} c_i^\dagger c_j + h_{xy} n_x n_y \right),$$

together with all its local truncations

$$H_\Lambda := \sum_{\{(x,y) \in \Lambda \mid |x-y|=1\}} \left(J_{xy} c_i^\dagger c_j + h_{xy} n_x n_y \right),$$

where the c_x and c_x^\dagger are Fermion annihilators and creators on the x -th site with the associated number operator $n_x := c_x^\dagger c_x$. The coupling constants J_{xy} and the external magnetic fields h_{xy} are independent random variables, and we suppose that the J_{xy} , as well as the h_{xy} , are identically distributed on a symmetric bounded interval of the real line according to the laws $J(s)$, $h(s)$, respectively. Denote $E\mathbb{Z}^d$ the edges of the standard lattice \mathbb{Z}^d . Our sample space (Ω, μ) for the pivotal model described above has the form

$$(7.2) \quad \Omega = \prod_{(x,y) \in E\mathbb{Z}^d} (\text{supp } J \times \text{supp } h), \quad d\mu = \prod_{(x,y) \in E\mathbb{Z}^d} (J(ds) \times h(ds)).$$

First of all notice that the shift α_x by $x \in \mathbb{Z}^d$ acts in a canonical way on the measurable space (Ω, μ) just by shifting the edges in the trajectories,

$$\omega = \{(x_\omega, y_\omega)\} \mapsto T_x \omega = \{(x_\omega + x, y_\omega + x)\}.$$

We have

Proposition 7.1. *Under the above notations, \mathbb{Z}^d acts on (Ω, μ) by a measure preserving mixing transformations.*

Proof. As the involved measure μ is a product of a single measure $J(ds) \times h(ds)$ and T_x is a bijection of Ω , it preserves μ . To check the ergodic properties of such an action, it is enough to reduce the matter to the measurable functions depending only by a finite number of variables. Let f, g be two of such functions. If $|x|$ is sufficiently big, f and g depend on different sets of variables $\Lambda_f, \Lambda_g + x$ in the space made of the edges of \mathbb{Z}^d . Then we get

$$\begin{aligned} \int f g \circ T_x d\mu &= \int f g \circ T_x d\mu_{\Lambda_f} \times d\mu_{\Lambda_g+x} = \int f d\mu_{\Lambda_f} \int g \circ T_x d\mu_{\Lambda_g+x} \\ &= \int f d\mu \int g \circ T_x d\mu = \int f d\mu \int g d\mu. \end{aligned}$$

□

Concerning the other useful regularity properties of the pivotal model described above, we need the following

Lemma 7.2. *If $A \in \text{CAR}(\Lambda)$ is localized in the bounded region Λ , then $f_{A,t} \in \text{CAR}(\bar{\Lambda}) \otimes L^\infty(\Omega, \mu)$, where $\bar{\Lambda} := \{x \in \mathbb{Z}^d \mid \text{dist}(x, \Lambda) \leq 1\}$.*

Proof. We have in this situation $\tau_t^\omega(A) = e^{iH_{\bar{\Lambda}}t} A e^{-iH_{\bar{\Lambda}}t}$. The proof follows by using the series expansion of the matrix $e^{iH_{\bar{\Lambda}}t}$. \square

Consider the map

$$(t, \omega) \in \mathbb{R} \times \Omega \mapsto \tau_t^\omega \in \text{Aut}(\text{CAR}(\mathbb{Z}^d)),$$

the last equipped with the σ -strong topology.¹⁵

Proposition 7.3. *The one parameter group of random automorphism τ_t^ω is jointly measurable in the variables (t, ω) .*

Proof. By taking into account the semimetrics (7.3) which generate the σ -strong topology, it is enough to check if all the functions $(t, \omega) \mapsto \varphi(\tau_t^\omega(A))$ are jointly measurable, when A and φ run over $\bigcup \text{CAR}(\Lambda)$ and $\bigcup \text{CAR}(\Lambda)^*$ respectively, where Λ are all the bounded subregions of \mathbb{Z}^d . As in Lemma 7.2 by expanding $e^{iH_{\bar{\Lambda}}t}$ in a power series, the functions mentioned above can be expressed as series whose the terms are measurable functions. \square

Proposition 7.4. *For each $A \in \text{CAR}(\mathbb{Z}^d)$, $f_{A,t} \in \text{CAR}(\mathbb{Z}^d) \otimes L^\infty(\Omega, \mu)$. In particular, $f_{A,t} \in L^\infty(\Omega, \mu; \text{CAR}(\mathbb{Z}^d))$*

Proof. Let $\{A_n\}_{n \in \mathbb{N}} \subset \bigcup_{\Lambda} \text{CAR}(\Lambda)$ be a sequence of localized elements converging to A . We get

$$\|f_{A,t} - f_{A_n,t}\|_\infty \leq \|A - A_n\|.$$

This means by Lemma 7.2, that $f_{A,t}$ is uniform limit of measurable functions belonging to $\text{CAR}(\mathbb{Z}^d) \otimes L^\infty(\Omega, \mu) \subset L^\infty(\Omega, \mu; \text{CAR}(\mathbb{Z}^d))$. The proof follows as $\text{CAR}(\mathbb{Z}^d) \otimes L^\infty(\Omega, \mu)$ is a closed subalgebra. \square

The pivotal model considered above represents the disordered version of the model considered for example in [39] (see also [37]), related to the investigation of the structure of the ground states. Concerning the temperature states, nothing is known regarding the possible existence of the critical temperature(s), even for the non disordered situation. Yet, it is possible to prove some general properties concerning the structure of the the KMS states. The reader is referred to [6]

¹⁵The two-sided uniform structure of the σ -strong topology is generated by the countable family if semimetrics

$$(7.3) \quad d_\varphi(\alpha, \beta) := \|\varphi \circ \alpha - \varphi \circ \beta\| + \|\varphi \circ \alpha^{-1} - \varphi \circ \beta^{-1}\|$$

where φ runs on countable dense subset of $\mathcal{S}(\mathcal{A})$.

for the non disordered situation.¹⁶ By taking into account the propositions 7.1, 7.3, 7.4, we can apply all the results of the present paper to the disordered model based on $\mathfrak{A} := \text{CAR}(\mathbb{Z}^d) \otimes L^\infty(\Omega, \mu)$ and the Hamiltonian (7.1), where (Ω, μ) is described by (7.2). We refer the reader to Section 5 of [9] for the proofs and details.

It is a well known fact (i.e. a standard compactness trick, see e.g. [15]) that, for a fixed realization of the couplings $\{J_{x,y}\}$, and the external magnetic field $\{h_{x,y}\}$, the spin algebra $\text{CAR}(\mathbb{Z}^d)$ admits KMS states at each inverse temperature $\beta > 0$. We start by considering in some detail the uniqueness case. Such a situation arises if the quantum model under consideration admits some critical temperature. The situation is well clarified for many classical disordered models (see e.g. [41]), contrary to the quantum situation where, to the knowledge of the authors, there are very few rigorous results concerning this point, even for the standard model of quantum spin glasses where the observables are modeled by the usual tensor product of infinitely many copies of a full matrix algebra. Namely, suppose that for a fixed $\beta > 0$, the Ising type model under consideration admits a unique KMS state, say φ_ω , almost surely. By the same arguments used in [9] we can show, thanks to Proposition 4.2, that the map $\omega \in \Omega \mapsto \varphi_\omega \in \mathcal{S}(\text{CAR}(\mathbb{Z}^d))$ is $*$ -weak measurable and made of even states almost surely. Furthermore, it satisfies almost surely the condition of equivariance

$$(7.4) \quad \varphi_\omega \circ \alpha_x = \varphi_{T_{-x}\omega}$$

w.r.t. the spatial translations, simultaneously. Namely, it defines by (4.5) a state $\varphi \in \mathcal{S}_{NI}(\mathfrak{A})$. Suppose now that ψ is any KMS state at the inverse temperature β , normal when restricted to $L^\infty(\Omega, \mu)$. Then, according to (4.5)

$$\psi = \int_{\Omega} \psi_\omega d\mu(\omega)$$

for a $*$ -weak measurable field $\omega \in \Omega \mapsto \psi_\omega \in \mathcal{S}(\text{CAR}(\mathbb{Z}^d))$ of positive form. By Proposition 4.4 and the uniqueness assumption, we get

$$\psi_\omega = \psi_\omega(\mathbf{1})\varphi_\omega,$$

almost surely. We have then shown that there exists a one-to-one correspondence $f \mapsto \varphi_f$ between positive normalized L^1 -functions and KMS states for \mathfrak{A} at inverse temperature $\beta > 0$. Namely,

$$(7.5) \quad \varphi(A) = \int_{\Omega} f(\omega)\varphi_\omega(A(\omega)) d\mu(\omega), \quad A \in \mathfrak{A},$$

¹⁶The reader is referred also to the papers [1, 24] for interesting connections which arise naturally between the Markov property for Fermions and the KMS condition and entanglement.

where $f \in L^1(\Omega, \mu)$ is any positive normalized function. In a situation such as the one just described above, there is a unique locally normal KMS state φ on \mathfrak{A} which is translation invariant, which correspond to $f = 1$ in (7.5), in addition, any KMS states is automatically even. Namely, $\varphi_f \in \mathcal{S}_N(\mathfrak{A})$. Finally, there exists a unique $\lambda > 0$ such that φ_f is a direct integral of III_λ factors almost surely.

We end the section by briefly describing what happens in “multiple phase” regime, provided such a possibility exists for the model under consideration. After taking the infinite-volume limit along various subsequences $\Lambda_{n_k} \uparrow \mathbb{Z}^d$, we will find, in general, different locally normal translation invariant \mathfrak{t} -KMS states on \mathfrak{A} at fixed inverse temperature β , which are automatically even. Fix one such a state φ . Then, one recovers a $*$ -weak measurable field $\{\varphi_\omega\}_{\omega \in \Omega} \subset \mathcal{S}(\text{CAR}(\mathbb{Z}^d))$ of even τ^ω -KMS states satisfying the equivariance property (7.4). According to Proposition 3.1 of [9] (cf. [15], Proposition 5.3.29), the set of the \mathfrak{t} -KMS states $\varphi_T \in \mathcal{S}(\mathfrak{A})$, locally normal w.r.t. φ , has the form

$$(7.6) \quad \varphi_T(A) = \int_{\Omega} \langle \pi_{\varphi_\omega}(A(\omega)) T(\omega)^{1/2} \Psi_{\varphi_\omega}, T(\omega)^{1/2} \Psi_{\varphi_\omega} \rangle_{\mathcal{H}_{\varphi_\omega}} d\mu(\omega).$$

Here, $(\pi_{\varphi_\omega}, \mathcal{H}_{\varphi_\omega}, \Psi_{\varphi_\omega})$ is the GNS representation of φ_ω , $\{T(\omega)\}_{\omega \in \Omega}$ is a measurable field of closed densely defined operators on $\mathcal{H}_{\varphi_\omega}$ affiliated to the (isomorphic) centres $\mathfrak{Z}_{\varphi_\omega}$ respectively, satisfying $\Psi_{\varphi_\omega} \in \mathcal{D}_{T(\omega)^{1/2}}$ almost surely, and $\int_{\Omega} \|T(\omega)^{1/2} \Psi_{\varphi_\omega}\|_{\mathcal{H}_{\varphi_\omega}}^2 d\mu(\omega) = 1$. This means that φ_T is the direct integral of

$$\varphi_{T(\omega)} := \langle \pi_{\varphi_\omega}(A(\omega)) T(\omega)^{1/2} \Psi_{\varphi_\omega}, T(\omega)^{1/2} \Psi_{\varphi_\omega} \rangle_{\mathcal{H}_{\varphi_\omega}}.$$

For physical application (cf. [5]), we specialize the situation when φ_T is even. Again by Proposition 4.2, this means that $\varphi_{T(\omega)}$ is even, almost surely. It might be proven that it implies that T is even, and then $T(\omega)$ is even almost surely. In order to avoid technicalities due to the unboundedness of T we prove the statement in the bounded case.

Proposition 7.5. *Suppose that the positive operator $T\eta\mathfrak{Z}_\varphi$, describing the even KMS state φ_T , which is normal w.r.t. φ , is bounded. Then $T(\omega)$ in (7.6) is even, almost surely.*

Proof. Suppose (\mathcal{B}, α) is a dynamical system, where \mathcal{B} is a \mathbb{Z}_2 -graded C^* -algebra and α is one parameter group of even automorphisms of \mathcal{B} . Let φ, ψ be even α -KMS states of \mathcal{B} with ψ normal w.r.t. φ . Consider the joint covariant (relative to the time evolution α_t , and the grading σ) GNS representation $(\mathcal{H}, \pi_\varphi, U_t, V, \Phi)$ of φ . By [15], Proposition 5.3.29

there exists a unique positive $T\eta\mathfrak{Z}_\varphi$, with $\Phi \in \mathcal{D}_{T^{1/2}}$ such that

$$\psi(A) = \langle \pi_\varphi(A)T^{1/2}\Phi, T^{1/2}\Phi \rangle.$$

Suppose now that T is bounded.¹⁷ As φ and ψ are even, we get for each $A \in \mathfrak{B}$.

$$\begin{aligned} \langle \pi_\varphi(A)\Phi, T\Phi \rangle &= \psi(A) = \psi(\sigma(A)) = \langle V\pi_\varphi(A)V\Phi, T\Phi \rangle \\ &= \langle \pi_\varphi(A)\Phi, VTV\Phi \rangle = \langle \pi_\varphi(A)\Phi, VTV\Phi \rangle. \end{aligned}$$

By the cyclicity of Φ we get $T\Phi = VTV\Phi$. As Φ is separating for \mathfrak{Z}_φ we conclude that $T = VTV$, that is T is even. In our situation, the fact that φ_T is supposed to be even and T bounded, implies that the $\varphi_{T(\omega)}$ are even and the $T(\omega)$ are bounded, almost surely. The proof follows by applying the previous consideration, fiberwise to $\varphi_{T(\omega)}$. \square

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¹⁷The proof can be generalized to the unbounded situation, approximating T by a sequence of bounded positive operators in \mathfrak{Z}_φ . We leave the details to the reader.

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